

Smoothing effects of dispersive equations on real rank one symmetric spaces

Koichi Kaizuka

Abstract

In this article we prove time-global smoothing effects of dispersive pseudodifferential equations with constant coefficient radially symmetric symbols on real rank one symmetric spaces of noncompact type. We also discuss gain of regularities according to decay rates of initial values for the Schrödinger evolution equation. We introduce some isometric operators and reduce the arguments to the well-known Euclidean case. In our proof, Helgason's Fourier transform and the Radon transform as an elliptic Fourier integral operator play crucial roles.

1 Introduction

Let X be a real rank one symmetric space of noncompact type. In this paper, we consider the initial value problem for dispersive equations of the form

$$D_t u - a(D_x)u = f(t, x) \quad \text{in } \mathbb{R} \times X, \quad (1.1)$$

$$u(0, x) = \varphi(x) \quad \text{in } X, \quad (1.2)$$

where $u(t, x)$ is a complex-valued unknown function of $(t, x) \in \mathbb{R} \times X$, $f(t, x)$ and $\varphi(x)$ are given functions, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $D_t = -i\partial_t$, and $a(D_x)$ is a pseudodifferential operator with a real-valued symbol on X (the precise definition shall be given later). In case that $a(D_x) = -\Delta_X$, where Δ_X is the Laplace-Beltrami operator on X , the equation (1.1) becomes the Schrödinger evolution equation.

The purpose of this paper is to study the smoothness of the solution to (1.1)-(1.2) with additional assumptions on the symbol of the pseudodifferential operator $a(D_x)$.

Roughly speaking, by introducing some isometric operators, we can translate all the time-global smoothing estimates on the one-dimensional Euclidean space into those on the real rank one symmetric space X .

2000 *Mathematics Subject Classification*. Primary 43A85; Secondary 35B65, 35S10.

Key words and phrases. Smoothing effect, microlocal analysis, symmetric space, Radon transform.

*The author is supported by the JSPS Research Fellowships for Young Scientists and the JSPS Grant-in-Aid for Scientific Research No.20-6882

We see the Radon transform \mathcal{R} as an elliptic Fourier integral operators, and recover the regularity of a function u on X from $\mathcal{R}u$ and the canonical relation of \mathcal{R}^* . Then we can also prove the gain of regularities for the solutions of the Schrödinger evolution equations on the real rank one symmetric space X from those on the one-dimensional Euclidean space.

First we will review some known results for dispersive equations on Euclidean spaces, mainly time-global spatially-local smoothing effects for real-principal-type pseudodifferential equations. We consider the initial value problem of the form

$$D_t u - a(D_x)u = F(t, x) \quad \text{in } \mathbb{R} \times \mathbb{R}^n, \quad (1.3)$$

$$u(0, x) = \phi(x) \quad \text{in } \mathbb{R}^n, \quad (1.4)$$

where $u(t, x)$ is a complex-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $F(t, x)$ and $\phi(x)$ are given functions, $a(\xi)$ is a real-valued function at most polynomial growth at infinity belongs to $C^1(\mathbb{R}^n)$ with $\nabla_\xi a(\xi) \neq 0$ for any $\xi \neq 0$, and the operator $a(D_x)$ is defined by

$$a(D_x)v(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\xi} a(\xi) v(y) dy d\xi$$

for an appropriate function v on \mathbb{R}^n .

Since $a(\xi)$ is real-valued, the initial value problem (1.3)-(1.4) is L^2 -well-posed, that is, for any $\phi \in L^2(\mathbb{R}^n)$ and for any $F \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^n))$, (1.3)-(1.4) possesses a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$.

In the case $a(\xi) = |\xi|^2$, i.e. $a(D_x) = -\Delta_{\mathbb{R}^n}$, the corresponding equation is the Schrödinger evolution equation and there exist many related works for its dispersive properties.

One of the origin of the studies for dispersive properties is the study for well-posedness of the initial value problem of the KdV type equations by Kato [23] around 1983.

In [24] Kato and Yajima obtained the following estimate in case $n \geq 3$:

$$\|\langle x \rangle^{-1} \langle D_x \rangle^{1/2} e^{-it\Delta_{\mathbb{R}^n}} \phi\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \leq C \|\phi\|_{L^2(\mathbb{R}^n)}.$$

Here we put $\langle x \rangle = (1 + |x|^2)^{1/2}$ and $\langle D_x \rangle = \mathcal{F}_{\mathbb{R}^n}^{-1} \langle \xi \rangle \mathcal{F}_{\mathbb{R}^n}$, $|D_x| = \mathcal{F}_{\mathbb{R}^n}^{-1} |\xi| \mathcal{F}_{\mathbb{R}^n}$.

In [1] Ben-Artzi and Klainerman also showed the following type estimate in case $n \geq 3$:

$$\|\langle x \rangle^{-\delta} |D_x|^{1/2} e^{-it\Delta_{\mathbb{R}^n}} \phi\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \leq C_\delta \|\phi\|_{L^2(\mathbb{R}^n)},$$

where $\delta > 1/2$ and $C_\delta > 0$ is a positive constant depending on δ .

In [30] Sugimoto obtained another type of time-global smoothing estimates for generalized Schrödinger operator not only for the homogeneous solutions but also for the inhomogeneous solutions.

Later in [22] Hoshiro obtained time-local smoothing effects for general real-principal polynomial symbols, that is, $a(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ ($m \geq 2, a_\alpha \in \mathbb{R}$) with its principal symbol $a_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ satisfies the dispersive condition $\nabla_\xi a_m \neq 0$ for any $\xi \neq 0$. Especially, in [22] Hoshiro proved that the dispersive condition $\nabla_\xi a_m(\xi) \neq 0$ ($\xi \neq 0$) is a necessary condition for the time-local spatially-local smoothing effects to hold.

Here we remark that the dispersive condition corresponding to a non-trapping condition for a Hamilton orbit generated by the principal symbol, i.e.

$$\nabla_\xi a_m(\xi) \neq 0 \quad (\xi \neq 0) \Leftrightarrow |x + t \nabla_\xi a_m(\xi)| \rightarrow \infty \quad (|t| \rightarrow \infty) \quad \text{for any } (x, \xi) \in T^*\mathbb{R}^n \setminus 0.$$

Recently, in [3] Chihara obtained time-global smoothing estimates for real-principal-type positively homogeneous symbols of degree $m > 1$, that is, real-valued functions $a(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap C^1(\mathbb{R}^n)$ such that $\nabla a(\xi) \neq 0$, $a(\xi) = |\xi|^m a(\xi/|\xi|)$ ($\xi \neq 0$). Set $p(\xi) = |\xi|^{(m-1)/2}$ if $n \geq 2$, $p(\xi) = a'(\xi)|\xi|^{-(m-1)/2}$ if $n = 1$. Then the estimate is as follows:

$$\|\langle x \rangle^{-\delta} p(D_x) u\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \leq C_\delta \left(\|\phi\|_{L^2(\mathbb{R}^n)} + \|\langle x \rangle^\delta |D_x|^{(m-1)/2} F\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \right),$$

where $\delta > 1/2$ and C_δ is a positive constant depending on δ .

Above results are all for equations with constant coefficients. The methods of the proofs are mainly based on some Fourier restriction theorem in the homogeneous case and the limiting absorption principle in the inhomogeneous case. Also, for equations with variable coefficients, there are so many results related to smoothing effects. For example, in [6] Doi proved that the non-trapping condition is necessary for the spatially-local smoothing effect of Schrödinger evolution groups on complete Riemannian manifolds. (Also see [4], [7], [8], [5], [3].)

Now we focus our interest on time-global smoothing effects for the Schrödinger evolution equation. Then above time-global estimates are essentially flat-Euclidean case. For other non-compact complete Riemannian manifolds time-global smoothing effects had not been obtained except for a special case. In [28] Rodnianski and Tao obtained a time-global smoothing estimate for the Schrödinger evolution equation on \mathbb{R}^3 with compact metric perturbations. From the geometrical point of view for the Schrödinger evolution equation, if we assume some “nice” geometric structures for the Riemannian manifold and the non-trapping condition for the geodesic flow, time-global smoothing effects may be expected. Moreover, to investigate the relation between geometrical conditions and the time-global smoothing effects is one of the interesting and deep problem.

In this paper Helgason’s Fourier transform plays crucial roles as in the Euclidean case. See section 2 for the notation and the rigorous definition for harmonic analysis on symmetric spaces.

In [15], [16] and [17] Helgason introduced the Fourier transform on symmetric spaces, and also proved the Plancherel formula, the inversion formula, and the Paley-Wiener theorem for his Fourier transform. After his pioneering works, harmonic analysis on symmetric spaces have been studied actively and applied to various fields of mathematics by many people.

On the symmetric space X of noncompact type, Helgason’s Fourier transform is defined by

$$\mathcal{F}u(\lambda, b) = \int_X e^{(-i\lambda + \rho)(A(x, b))} u(x) dx \quad \text{for } u \in \mathcal{D}(X)$$

for $(\lambda, b) \in \mathfrak{a}^* \times B$. And the Fourier transform is invertible by the following formula:

$$u(x) = \iint_{\mathfrak{a}_+^* \times B} e^{(i\lambda + \rho)(A(x, b))} \mathcal{F}u(\lambda, b) |\mathfrak{c}(\lambda)|^{-2} d\lambda db \quad \text{for any } x \in X.$$

In the rest of this section, we will state our main results.

For any real-valued function $a(\lambda) \in C(\overline{\mathfrak{a}_+^*}) \cap C^1(\mathfrak{a}_+^*)$ at most polynomial order at infinity, the pseudodifferential operator $a(D_x)$ is defined by

$$a(D_x)v(x) = \iint_{\mathfrak{a}_+^* \times B} e^{(i\lambda + \rho)(A(x, b))} a(\lambda) \mathcal{F}v(\lambda, b) |\mathbf{c}(\lambda)|^{-2} d\lambda db.$$

for an appropriate function v on X .

Since $a(\lambda)$ is real-valued, the initial value problem (1.1)-(1.2) is L^2 -well-posed, that is, for any $\varphi \in L^2(X, dx)$ (abbreviated $L^2(X)$ in the sequel) and for any $f \in L_{\text{loc}}^1(\mathbb{R}; L^2(X))$, (1.1)-(1.2) possesses a unique solution $u \in C(\mathbb{R}; L^2(X))$. Moreover, the unique solution u is explicitly given by

$$\begin{aligned} u(t, x) &= e^{ita(D_x)}\varphi(x) + iGf(t, x), \\ e^{ita(D_x)}\varphi(x) &= \iint_{\mathfrak{a}_+^* \times B} e^{(i\lambda + \rho)(A(x, b))} e^{ita(\lambda)} \mathcal{F}\varphi(\lambda, b) |\mathbf{c}(\lambda)|^{-2} d\lambda db, \\ Gf(t, x) &= \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) d\tau. \end{aligned}$$

Here we introduce a “weighted” L^2 -space on X as follows.

Definition 1.1. Let $\delta \in \mathbb{R}$, and for all $u, v \in \mathcal{D}(X)$, define an inner metric

$$(u, v)_{L^{2, \delta}(X)} = w^{-1} \iint_{\mathfrak{a}^* \times B} \langle D_\lambda \rangle^\delta (\mathcal{F}u(\lambda, b) \mathbf{c}^{-1}(\lambda)) \overline{\langle D_\lambda \rangle^\delta (\mathcal{F}v(\lambda, b) \mathbf{c}^{-1}(\lambda))} d\lambda db,$$

and “weighted” L^2 -norm $\|u\|_{L^{2, \delta}(X)} = (u, u)_{L^{2, \delta}(X)}^{1/2}$. Let $L^{2, \delta}(X)$ be the completion of the pre-Hilbert space $(\mathcal{D}(X), (\cdot, \cdot)_{L^{2, \delta}(X)})$ (well-definedness will be discussed in section 3).

In what follows we assume that X is real rank one. From Tits’ classification [31] of isotropic homogeneous manifolds, the real rank one symmetric spaces of noncompact type are exactly the following four types of noncompact complete Riemannian manifolds,

$$H^n(\mathbb{R}), H^n(\mathbb{C}), H^n(\mathbb{H}), \text{ or } H^2(\mathbb{O}),$$

those are, the real, complex, quaternion hyperbolic space or the octave hyperbolic plane respectively, where \mathbb{H} is the quaternion and \mathbb{O} is the octave or called the Cayley algebra.

Here we state our first main results.

Theorem 1.2. (i) Let $a(\lambda) \in C^1(\mathfrak{a}^*)$ and $p(\lambda) \in C^0(\mathfrak{a}^*)$ be real-valued functions at most polynomial order at infinity. Suppose that the pseudodifferential operator $a(D_H)$ on \mathfrak{a} causes a time-global estimate for homogeneous solutions, that is, there exist positive constants $\delta > 1/2$ and C_δ such that

$$\|\langle H \rangle^{-\delta} p(D_H) e^{ita(D_H)} \phi\|_{L^2(\mathbb{R} \times \mathfrak{a}, dt dH)} \leq C_\delta \|\phi\|_{L^2(\mathfrak{a}, dH)}.$$

Then we have

$$\|p(D_x)e^{ita(D_x)}\varphi\|_{L^2(\mathbb{R};L^{2,-\delta}(X))} \leq w^{1/2}C_\delta\|\varphi\|_{L^2(X)}.$$

(ii) Let $a(\lambda) \in C^1(\mathfrak{a}^*)$ and $q(\lambda) \in C^0(\mathfrak{a}^*)$ be real-valued functions at most polynomial order at infinity. Suppose that the pseudodifferential operator $a(D_H)$ on \mathfrak{a} causes a time-global estimate for inhomogeneous solutions, that is, there exist positive constants $\delta > 1/2$, C_δ and a cut off function $\chi \in C^\infty(\mathfrak{a}^*)$ with $\chi(\lambda) = 0$ ($|\lambda| \leq 1$), $\chi(\lambda) = 1$ ($|\lambda| \geq 2$) such that

$$\left\| \langle H \rangle^{-\delta} \chi(D_H) q(D_H) \int_0^t e^{i(t-\tau)a(D_H)} F(\tau) d\tau \right\|_{L^2(\mathbb{R} \times \mathfrak{a}, dt dH)} \leq C_\delta \|\langle H \rangle^\delta F\|_{L^2(\mathbb{R} \times \mathfrak{a}, dt dH)},$$

Then there exists a positive constant $C_{\delta,\chi}$ such that

$$\left\| \chi(D_x) q(D_x) \int_0^t e^{i(t-\tau)a(D_x)} f(\tau) d\tau \right\|_{L^2(\mathbb{R};L^{2,-\delta}(X))} \leq C_{\delta,\chi} \|f\|_{L^2(\mathbb{R};L^{2,\delta}(X))}.$$

Corollary 1.3. Let $a(D_x)$ be a polynomial of the Laplace-Beltrami operator Δ_X of real coefficients, then for any $\delta > 1/2$ and $\chi \in C^\infty(\mathfrak{a}^*)$ as in Theorem 1.2 there exist positive constants $C_\delta, C_{\delta,\chi}$ such that

$$\begin{aligned} \| |a'(D_x)|^{1/2} e^{ita(D_x)} \varphi \|_{L^2(\mathbb{R};L^{2,-\delta}(X))} &\leq C_\delta \|\varphi\|_{L^2(X)}, \\ \left\| \chi(D_x) a'(D_x) \int_0^t e^{i(t-\tau)a(D_x)} f(\tau) d\tau \right\|_{L^2(\mathbb{R};L^{2,-\delta}(X))} &\leq C_{\delta,\chi} \|f\|_{L^2(\mathbb{R};L^{2,\delta}(X))}. \end{aligned}$$

Remark. Needless to say, the estimates above are meaningless without some regularity estimates for the “weighted” L^2 -space $L^{2,\delta}(X)$ and we will study local regularities and basic properties of $L^{2,\delta}(X)$ in Section 3.

Here we prepare standard notation for micorolocal analysis.

Definition 1.4. Let M be an n -dimensional smooth manifold and s be any real number. A distribution $v \in \mathcal{D}'(M)$ is in the local Sobolev space $H_{\text{loc}}^s(M)$ if and only if $\rho^*(\chi v) \in H^s(\mathbb{R}^n)$ for any local chart (ρ, U) and cut-off function $\chi \in C_0^\infty(U)$. Also the distribution $v \in \mathcal{D}'(M)$ is in H^s microlocally near $(x_0, \xi_0) \in T^*M \setminus 0$ if and only if there exists a local chart (ρ, U) , cut-off function $\chi \in C_0^\infty(U)$ with $\chi(x_0) \neq 0$ and a conic neighborhood $V \subset \mathbb{R}^n \setminus \{0\}$ of $\eta_0 = (\rho^{-1})^*\xi_0$ such that $\langle \eta \rangle^s \mathcal{F}_{\mathbb{R}^n}(\rho^*(\chi v))(\eta) \in L^2(V, d\eta)$. Let $\text{WF}^s(v)$ denote the elements of $T^*M \setminus 0$ such that v is not in H^s microlocally. $\text{WF}^s(v)$ is called the Sobolev wave front set of $v \in \mathcal{D}'(M)$ of order $s \in \mathbb{R}$.

Also, we have the gain of regularity for the Schrödinger evolution equation as following.

Theorem 1.5. Suppose $\varphi \in L^2(X)$ satisfies the following condition for some positive integer k and open set $\Theta \subset B$:

$$\|\langle D_\lambda \rangle^k \{ \mathcal{F}\varphi(\lambda, b) \mathbf{c}(\lambda)^{-1} \} \|_{L^2(\mathfrak{a}^* \times \Theta, d\lambda db)} < \infty. \quad (1.5)$$

Then for all $x \in X$, and $b \in \Theta_x$ we have

$$\begin{aligned} (x; \pm\omega_b(x)) &\notin \text{WF}^{k+1/2}(e^{-it\Delta}\varphi) \quad \text{for a.e. } t \neq 0, \\ (x; \pm\omega_b(x)) &\notin \text{WF}^k(e^{-it\Delta}\varphi) \quad \text{for any } t \neq 0. \end{aligned}$$

In particular, if $\phi \in L^{2,k}(X)$, then we have

$$\begin{aligned} e^{-it\Delta_x}\varphi &\in H_{\text{loc}}^{k+1/2}(X) \quad \text{for a.e. } t \neq 0, \\ e^{-it\Delta_x}\varphi &\in H_{\text{loc}}^k(X) \quad \text{for any } t \neq 0. \end{aligned}$$

Here $\omega_b(x) = d_x(A(x, b)) \in \Lambda^1(X)$, $\Theta_{g \circ} = \Theta \cap (g \circ (-\text{id}_B) \circ g^{-1}\Theta)$, $g \in G$.

Theorem 1.6. For any $k \in \mathbb{N}$ and $\delta > 1/2$, we have the following continuous maps:

$$\begin{aligned} L^{2,k}(X) \ni \varphi &\mapsto t^k \langle D_x \rangle^{k+1/2} e^{-it\Delta_x} \varphi \in L_{\text{loc}}^2(\mathbb{R}; L^{2,-k-\delta}(X)), \\ L^{2,k}(X) \ni \varphi &\mapsto t^k \langle D_x \rangle^k e^{-it\Delta_x} \varphi \in C(\mathbb{R}; L^{2,-k}(X)). \end{aligned}$$

Remark. (i) The results in Theorem 1.5 and 1.6 may be contained in Doi's results in [7], but our proof takes different approach and is rather simple. So we will treat those as theorems.

(ii) Here we remark that the condition (1.5) corresponds to a decaying along some family of geodesics. In fact, we can rewrite the norm by using a pseudodifferential operator Λ and the Radon transform \mathcal{R} as following (Λ and \mathcal{R} are given in the next section):

$$\left\| \langle D_\lambda \rangle^k \{ \mathcal{F}\varphi(\lambda, b) c(\lambda)^{-1} \} \right\|_{L^2(\mathfrak{a}^* \times \Theta, d\lambda db)} = \left\| \langle H \rangle^k \Lambda \mathcal{R}\varphi \right\|_{L^2(\mathfrak{a} \times \Theta, d\xi(H, b))}.$$

(iii) We can restate the results in Theorem 1.5 by using geodesics: For a maximal geodesic $\gamma : \mathbb{R} \rightarrow X$, if $b_+ = \gamma(+\infty)$, $b_- = \gamma(-\infty) \in \Theta$, then for any $s \in \mathbb{R}$ we have

$$\begin{aligned} (\gamma(s); \omega_{b_\pm}(\gamma(s))) &\notin \text{WF}^{k+1/2}(e^{-it\Delta}\varphi) \quad \text{for a.e. } t \neq 0, \\ (\gamma(s); \omega_{b_\pm}(\gamma(s))) &\notin \text{WF}^k(e^{-it\Delta}\varphi) \quad \text{for any } t \neq 0. \end{aligned}$$

Finally, this paper is organized as follows. In Section 2 we review Helgason's harmonic analysis on symmetric spaces. In Section 3 we prove local regularities for elements of weighted L^2 -space. In Section 4 we prove Theorem 1.2. In Section 5 we prove Theorems 1.5 and 1.6.

Acknowledgments

The author would like to express his sincere gratitude to Hiroyuki Chihara for his great encouragement, valuable and precise advices. He would like to thank the members of the seminar, Ryuichiro Mizuhara and Eiji Onodera for their kindly advices and helpful discussions.

2 Preliminaries

In this section we introduce Helgason's harmonic analysis and prepare some lemmas needed later. One can consult with Helgason's books [18], [19] and [20] on the basic facts on harmonic analysis on symmetric space. Also, we will establish some tools on symmetric spaces, related to the proof of the theorems.

In this section, X is an arbitrary rank symmetric space unless we impose some conditions in addition.

Let X be a Riemannian symmetric space of noncompact type, that is, $X = G/K$ where G is a noncompact, connected, semisimple Lie group with finite center and K is a maximal compact subgroup of G . Let \mathfrak{g} and \mathfrak{k} be the Lie algebra of G and K respectively. Let B be the Killing form on \mathfrak{g} , that is, $B(X, Y) = \text{Trace}(\text{ad}(X) \text{ad}(Y))$ for $X, Y \in \mathfrak{g}$, where $\text{ad}(X)Y = [X, Y]$, and θ be the Cartan involution associated with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\langle X, Y \rangle = -B(X, \theta Y)$, then $\langle \cdot, \cdot \rangle$ defines an inner metric on \mathfrak{g} . Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and \mathfrak{a}^* its dual. Then $l = \dim \mathfrak{a}$ is called the real rank of G , or X . For $\alpha \in \mathfrak{a}^*$, put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}; [H, X] = \alpha(H)X, H \in \mathfrak{a}\}$$

and $m_\alpha = \dim \mathfrak{g}_\alpha$. If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, then α is called a restricted root and denotes all restricted roots by Σ . Let $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$ denote the complexification of \mathfrak{g} and \mathfrak{a}^* , respectively. If $\lambda, \mu \in \mathfrak{a}_\mathbb{C}^*$, let $H_\lambda \in \mathfrak{a}_\mathbb{C}$ be determined by $\lambda(H) = \langle H_\lambda, H \rangle$ for $H \in \mathfrak{a}$ and put $\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle$. Since B is positive definite on \mathfrak{p} , we put $|\lambda| = \langle \lambda, \lambda \rangle^{1/2}$ for $\lambda \in \mathfrak{a}^*$ and $|X| = \langle X, X \rangle^{1/2}$ for $X \in \mathfrak{p}$. Then the natural identification $(D\pi_G)_e|_{\mathfrak{p}} : \mathfrak{p} \rightarrow T_o X$ (denote $o = eK$) induces the left G -invariant Riemannian metric on X . Let dx be the Riemannian measure, and Δ_X be the Laplace-Beltrami operator on $X = G/K$. Let \mathfrak{a}' be the open subset of \mathfrak{a} where all the restricted roots are $\neq 0$. Fix a Weyl chamber \mathfrak{a}^+ in \mathfrak{a}' that is a connected component of \mathfrak{a}' , and call $\alpha \in \Sigma$ is positive if it is positive on \mathfrak{a}^+ . Let Σ^+ denotes the set of all positive roots. Put $\rho = (1/2) \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$, $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ and let N denote the corresponding analytic subgroup of G . Then we obtain an Iwasawa decomposition $G = KAN$ of G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ of \mathfrak{g} . Each $g \in G$ can be uniquely written $g = \kappa(g) \exp(H(g))n(g)$, by $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$, and $n(g) \in N$. Let M denote the centralizer of A in K , M' the normalizer of A in K , and the W factor group M'/M , called the Weyl group. The group W acts as a group of linear transformations of \mathfrak{a}^* by $(s\lambda)(H) = \lambda(s^{-1} \cdot H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}^*$ and $s \in W$, where $g \cdot X = \text{Ad}(g)X$ for $g \in G$, $X \in \mathfrak{g}$. Let w denote the order of W . We fix an orthonormal basis on \mathfrak{a} with respect to the Killing form and its dual basis on \mathfrak{a} , then we can regard \mathfrak{a} and \mathfrak{a}^* as the Euclidean spaces of dimension l respectively. The Killing form induces the Euclidean measures on \mathfrak{a} and \mathfrak{a}^* , multiplying these by the factor $(2\pi)^{-l/2}$ we obtain the invariant measures dH and $d\lambda$ on \mathfrak{a} and \mathfrak{a}^* respectively. Put $B = K/M = G/MAN$. Let \mathfrak{m} be the Lie algebra of M and \mathfrak{l} be the orthogonal complement of \mathfrak{m} in \mathfrak{k} with respect to the Killing form. Since $\langle \cdot, \cdot \rangle$ is strictly positive on \mathfrak{l} , the natural identification $(D\pi_K)_e|_{\mathfrak{l}} : \mathfrak{l} \rightarrow T_{b_0} B$ (denote $b_0 = eM$) induces the left K -invariant Riemannian metric on B . Let db be the left K -invariant measure on B normalized so that total mass equals one. Put $\Xi = G/MN$, called the horocycle space of X , and let $d\xi$ be the invariant measure on Ξ . We can naturally identify Ξ with $\mathfrak{a} \times K/M$ by the diffeomorphism $\mathfrak{a} \times K/M \ni (H, kM) \mapsto k \cdot \exp(H)MN \in \Xi$, then we can write $d\xi = e^{2\rho(H)} dH db$. For

$x \in X, b \in B$ let $\xi(x, b)$ be the horocycle passing through the point $x = g \cdot o \in X$ with normal $b = kM \in B$, and let $A(x, b) = -H(g^{-1}k) \in \mathfrak{a}$ be the composite distance from the origin to $\xi(x, b)$. (See Figure 1.)

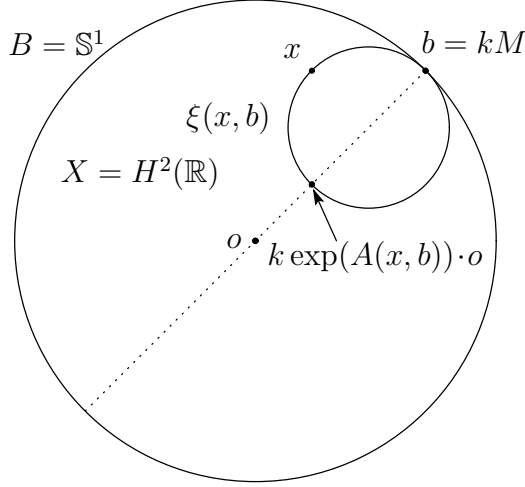


Figure 1: The horocycle on the Poincaré disc

Now, we have the following fundamental identity

$$\Delta_X(e^{(-i\lambda+\rho)(A(x,b))}) = -(|\lambda|^2 + |\rho|^2)e^{(-i\lambda+\rho)(A(x,b))},$$

then the Helgason Fourier transform of $f \in \mathcal{D}(X)$ is given by

$$\mathcal{F}f(\lambda, b) = \int_X e^{(-i\lambda+\rho)(A(x,b))} f(x) dx, \quad (\lambda, b) \in \mathfrak{a}_{\mathbb{C}}^* \times B.$$

Helgason proved the inversion formula and the Plancherel theorem as follows:

Theorem 2.1. *We have the followings:*

- **Inversion formula** ; For $f \in \mathcal{D}(X)$, the Fourier transform is inverted by

$$f(x) = w^{-1} \iint_{\mathfrak{a}^* \times B} e^{(i\lambda+\rho)(A(x,b))} \mathcal{F}f(\lambda, b) |\mathbf{c}(\lambda)|^{-2} d\lambda db \quad \text{for all } x \in X,$$

where \mathbf{c} is Harish-Chandra's \mathbf{c} -function.

- **The Plancherel theorem** ; The Fourier transform is extended to the unitary isomorphism

$$\mathcal{F} : L^2(X, dx) \rightarrow L_W^2(\mathfrak{a}^* \times B, w^{-1} |\mathbf{c}(\lambda)|^{-2} d\lambda db),$$

where $L_W^2(\mathfrak{a}^* \times B, |\mathbf{c}(\lambda)|^{-2} d\lambda db)$ consisting of all elements of $L^2(\mathfrak{a}^* \times B, |\mathbf{c}(\lambda)|^{-2} d\lambda db)$ satisfying the following condition

$$\int_B e^{(is\lambda+\rho)(A(x,b))} \psi(s\lambda, b) db = \int_B e^{(i\lambda+\rho)(A(x,b))} \psi(\lambda, b) db \quad (2.1)$$

for all $s \in W$, and a.e. $x \in X, b \in B$.

Remark. By the W -invariantness of the Fourier images, we can rewrite the inversion formula as follows:

$$f(x) = \iint_{\mathfrak{a}_+^* \times B} e^{(i\lambda + \rho)(A(x, b))} \mathcal{F}f(\lambda, b) |\mathbf{c}(\lambda)|^{-2} d\lambda db.$$

Also we have the following unitary isomorphism:

$$\mathcal{F} : L^2(X, dx) \rightarrow L^2(\mathfrak{a}_+^* \times B, |\mathbf{c}(\lambda)|^{-2} d\lambda db).$$

Next, we introduce the Schwartz space $\mathcal{S}(X)$ defined by Eguchi and Okamoto in [9]. They characterized the image of it by the Fourier transform. Here we identify the functions on $X = G/K$ with the right K -invariant functions on G . For a function f on X , and for $X, X' \in \mathfrak{g}$ we put

$$\begin{aligned} f(X; g) &= \left. \frac{d}{dt} f(\exp(tX)g) \right|_{t=0}, \\ f(g; X') &= \left. \frac{d}{dt} f(g \exp(tX')) \right|_{t=0}. \end{aligned}$$

And let $U(\mathfrak{g}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, then for X and $X' \in U(\mathfrak{g}_{\mathbb{C}})$, $f(X; g)$ and $f(g; X')$ are naturally defined respectively as the homomorphic extension of the above definition.

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, let $\varphi_\lambda(x) \in \mathcal{E}(X)$ be the elementary spherical function, that is,

$$\varphi_\lambda(x) = \int_B e^{(i\lambda + \rho)(A(x, b))} db,$$

then $\varphi_\lambda(x)$ is a K -invariant eigenfunction on X . For $\varphi_0(x)$, following basic properties are known. (For the detail, see [10], [14].)

Proposition 2.2. *We have*

(i) *There exist a positive constant C_1 , and a positive integer $N_1 \in \mathbb{N}$ such that*

$$e^{-\rho(\log a)} \leq \varphi_0(a) \leq C_1 e^{-\rho(\log a)} (1 + \sigma(a))^{N_1}$$

for any $a \in A^+ = \exp \mathfrak{a}^+$.

(ii) *There exists a positive integer $N_2 \in \mathbb{N}$ such that*

$$\int_G \varphi_0(g)^2 (1 + \sigma(g))^{-N_2} dg < \infty.$$

Now we define the Schwartz space on the symmetric space X .

Definition 2.3. Let $\mathcal{S}(X)$ denote the set of C^∞ functions f on $X = G/K$ satisfying

$$\tau_{N,X,X'}(f) = \sup_{g \in G} \{ |f(X; g; X')| \varphi_0(g)^{-1} (1 + \sigma(g))^N \} < \infty,$$

for any $N \in \mathbb{Z}_{\geq 0}$, and $X, X' \in U(\mathfrak{g}_{\mathbb{C}})$. Then $\mathcal{S}(X)$ becomes a Fréchet space with the seminorms $\tau_{N,X,X'}$ ($N \in \mathbb{Z}_{\geq 0}$, $X, X' \in U(\mathfrak{g}_{\mathbb{C}})$). We can easily see that

$$\mathcal{D}(X) \hookrightarrow \mathcal{S}(X) \hookrightarrow L^2(X),$$

where all inclusions are continuous and dense embeddings.

We also define the Schwartz space on the “phase space” $\mathfrak{a}^* \times B$.

Definition 2.4. Let $\mathcal{S}(\mathfrak{a} \times B)$ denote the set of C^∞ functions ψ on $\mathfrak{a} \times B$ which satisfying

$$\nu_{N,\alpha,m} = \sup_{\mathfrak{a}^* \times B} \{ |(\partial_\lambda^\alpha \Delta_B^m \psi)(\lambda, b)| (1 + |\lambda|)^N \} < \infty,$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^l$, $m, N \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{S}_W(\mathfrak{a}^* \times B)$ denote all the W -invariant elements ψ in $\mathcal{S}(\mathfrak{a}^* \times B)$, i.e. $\psi \in \mathcal{S}(\mathfrak{a}^* \times B)$ and satisfying

$$\int_B e^{(is\lambda + \rho)(A(x,b))} \psi(s\lambda, b) db = \int_B e^{(i\lambda + \rho)(A(x,b))} \psi(\lambda, b) db$$

for all $s \in W$, $\lambda \in \mathfrak{a}^*$, and $x \in X$. Then both $\mathcal{S}(\mathfrak{a}^* \times B)$ and $\mathcal{S}_W(\mathfrak{a}^* \times B)$ are Fréchet spaces with the seminorms $\nu_{N,\alpha,m}$ ($\alpha \in \mathbb{Z}_{\geq 0}^l$, $m, N \in \mathbb{Z}_{\geq 0}$).

In [9], Eguchi and Okamoto proved the following theorem.

Theorem 2.5 (M. Eguchi and K. Okamoto). *The Fourier transform \mathcal{F} is a linear topological isomorphism of $\mathcal{S}(X)$ onto $\mathcal{S}_W(\mathfrak{a}^* \times B)$.*

Next, we will see some basic properties of Harish-Chandra’s c -function and prove a key lemma to apply pseudodifferential calculi on \mathfrak{a} and \mathfrak{a}^* in the later section.

Let Σ_0 be the indivisible roots of Σ , that is, whose elements are $\alpha \in \Sigma$ so that $\alpha/2 \notin \Sigma$, and put $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$, called positive indivisible roots. Then Harish-Chandra’s c -function can be extended to the meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ and is explicitly given by as follows.

Theorem 2.6 (The Gindikin-Karpelevič formula). *The c -function for the semisimple Lie group G is given by the absolutely convergent integral*

$$c(\lambda) = \int_{\bar{N}} e^{-(i\lambda + \rho)(H(\bar{n}))} d\bar{n}, \quad \operatorname{Re}(i\lambda) \in \mathfrak{a}_+^*,$$

where $\bar{N} = \theta N$, and $d\bar{n}$ is the Haar measure on \bar{N} normalized by $\int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1$.

Also, $c(\lambda)$ is explicitly given by the formula

$$c(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{-\langle i\lambda, \alpha_0 \rangle} \Gamma(\langle i\lambda, \alpha_0 \rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0 \rangle)) \Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle))},$$

where $\alpha_0 = \alpha / \langle \alpha, \alpha \rangle$, the constant c_0 is given so that $c(-i\rho) = 1$, and $\Gamma(z)$ is the gamma function.

Remark. We have

- (i) Zeros of $c(\lambda)^{-1}$ on \mathfrak{a}^* is precisely the Weyl walls, i.e. $\bigcup_{\alpha \in \Sigma_0^+} \ker H_\alpha$.
- (ii) $|c(\lambda)|^{-2} = c(s\lambda)c(-s\lambda)$ for all $s \in W, \lambda \in \mathfrak{a}^*$.
- (iii) If $\lambda, \mu \in \Sigma$ are proportional, i.e. $\mu = c\lambda$ for some $c \in \mathbb{R}$, then $c = \pm 1/2, \pm 1$, or ± 2 .
(See [20, Chapter X, Section 3].)

In the following, we will use the pseudodifferential operator theory freely. See e.g. [21], [25], and [29] for the detail.

A key lemma in this paper is the following.

Lemma 2.7. *We have*

- (i) $c^{-1}(\lambda) \in S^{\dim N/2}$, i.e. for any $\alpha \in \mathbb{Z}_{\geq 0}^l$ there exists a positive constant $C_\alpha > 0$ such that

$$|\partial_\lambda^\alpha c^{-1}(\lambda)| \leq C_\alpha \langle \lambda \rangle^{\dim N/2 - |\alpha|}$$

uniformly on \mathfrak{a}^* .

- (ii) If X is real rank one, then $c^{-1}(\lambda)$ becomes an elliptic symbol of order $\dim N/2$.

Proof. Our proof is almost all same as in [19, Lemma 3.6, p.110], but we need a little bit more precise estimates for $c^{-1}(\lambda)$ to use pseudodifferential calculus.

- (i) For the gamma function following formulas are well known:

$$\begin{aligned} \Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) &= 2^{1-z}\sqrt{\pi}\Gamma(z) \quad \text{on } \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, \\ \Gamma(z+1) &= z\Gamma(z) \quad \text{on } \mathbb{C} \setminus \mathbb{Z}_{\leq 0}. \end{aligned}$$

By using these formula, we can write $c^{-1}(\lambda)$ as the following form

$$\begin{aligned} c^{-1}(\lambda) &= c_0^{-1} \left(\prod_{\alpha \in \Sigma_0^+} 2\sqrt{\pi} \langle i\lambda, \alpha_0 \rangle \right) \\ &\times \left(\prod_{\alpha \in \Sigma_0^+} s\left(\frac{\langle \lambda, \alpha_0 \rangle}{2}; \frac{1}{4}m_\alpha + \frac{1}{2}, \frac{1}{2}\right) s\left(\frac{\langle \lambda, \alpha_0 \rangle}{2}; \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha}, 1\right) \right) \end{aligned} \quad (2.2)$$

where we put $s(\xi; a, b) = \Gamma(a + i\xi)/\Gamma(b + i\xi)$ for $\xi \in \mathbb{R}$ and $a, b > 0$. So we first investigate $s(\xi; a, b)$. Using the formula for the gamma function

$$\frac{\Gamma'(z)}{\Gamma(z)} = \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+z} \right) - \gamma - \frac{1}{z},$$

where $\gamma = \lim_{m \rightarrow \infty} \{(1 + \frac{1}{2} + \cdots + \frac{1}{m}) - \log m\}$ is the Euler constant, we can find that

$$\begin{aligned} -i \frac{s'(\xi; a, b)}{s(\xi; a, b)} &= \frac{\Gamma'(a + i\xi)}{\Gamma(a + i\xi)} - \frac{\Gamma'(b + i\xi)}{\Gamma(b + i\xi)} \\ &= \sum_{m=0}^{\infty} \frac{(a-b)}{(m+a+i\xi)(m+b+i\xi)}. \end{aligned}$$

So we put

$$t(\xi; a, b) = i \sum_{m=0}^{\infty} \frac{(a-b)}{(m+a+i\xi)(m+b+i\xi)},$$

then we have

$$s'(\xi; a, b) = s(\xi; a, b)t(\xi; a, b). \quad (2.3)$$

By an elementary calculation, we get $t(\xi; a, b) \in S^{-1}$, so if we can verify an asymptotic behavior of $s(\xi; a, b)$ as $|\xi| \rightarrow \infty$, then we obtain its derivatives of $s(\xi; a, b)$. Now we use the formula for the gamma function

$$\lim_{|z| \rightarrow \infty} \frac{\Gamma(z+c)}{\Gamma(z)} e^{-c \log z} = 1,$$

where $|\arg z| \leq \pi - \delta$, $0 < \delta < \pi$, and $c \in \mathbb{C}$. Since we can write

$$s(\xi; a, b) = \frac{\Gamma(a+i\xi)}{\Gamma(i\xi)} \frac{\Gamma(i\xi)}{\Gamma(b+i\xi)}$$

and $|e^{-(a-b) \log(i\xi)}| = |\xi|^{-(a-b)}$, we get

$$\lim_{|\xi| \rightarrow \infty} |s(\xi; a, b)| |\xi|^{-(a-b)} = 1. \quad (2.4)$$

Hence by (2.2), (2.3), and $t(\xi; a, b) \in S^{-1}$, we obtain $s(\xi; a, b) \in S^{(a-b)}$. Since $\langle \lambda, \alpha_0 \rangle$ is a polynomial of $\lambda \in \mathfrak{a}^*$ of order one, by using expression (2.2) we have $c^{-1}(\lambda) \in S^m$. By a simple calculation, we have $m = \dim N/2$.

(ii) If X is real rank one, then $|\langle i\lambda, \alpha_0 \rangle| = |\lambda|$ for all $\lambda, \alpha \in \mathfrak{a}^*$, so ellipticity follows from (2.2) and (2.4). \square

Finally, we shall introduce the Radon transform and see its microlocal properties.

For $f \in \mathcal{D}(X)$, the Radon transform is defined by

$$\mathcal{R}f(\xi) = \int_{x \in \xi} f(x) dm_\xi(x) \quad \text{for } \xi \in \Xi,$$

where dm_ξ denotes the induced measure on ξ , or equivalently under the identification $\mathfrak{a} \times B \ni (H, b) \leftrightarrow \xi(H, b) \in \Xi$, we can also write

$$\mathcal{R}f(H, kM) = \int_N f(k \exp(H)n \cdot o) dn.$$

For $\varphi \in L^1_{\text{loc}}(\Xi, d\xi)$, the dual Radon transform is defined by

$$\mathcal{R}^*\varphi(x) = \int_{\xi \ni x} \varphi(\xi) dm_x(\xi) \quad \text{for } x \in X,$$

where dm_x denotes the induced measure on $\{\xi \in \Xi; \xi \ni x\}$. Then \mathcal{R}^* is the formal adjoint of \mathcal{R} in the following sense:

$$\int_X f(x) \mathcal{R}^* \varphi(x) dx = \int_{\Xi} \mathcal{R} f(\xi) \varphi(\xi) d\xi, \quad f \in \mathcal{D}(X), \varphi \in \mathcal{E}(\Xi).$$

For the Radon transform, the following inversion formula holds. (For the detail and the proof, see e.g. [19].)

Theorem 2.8. *For any $u \in \mathcal{S}(X)$, the Radon transform is invertible by*

$$u = w^{-1} \mathcal{R}^* \bar{\Lambda} \Lambda \mathcal{R} \quad \text{on } X.$$

Where Λ is a pseudodifferential operator on Ξ defined by

$$\Lambda = e^{-\rho(H)} \mathbf{c}^{-1} (D_H) e^{\rho(H)}.$$

Next, we will see microlocal properties of the Radon transform and establish a key theorem.

In [12] Guillemin and Sternberg introduced microlocal techniques to the study of generalized Radon transform between two manifolds X, Y by noting that δ_Z , where $Z \subset X \times Y$ is a submanifold called incidence relation, is an example of a Fourier integral distribution and proceeding to study the microlocal analogue of the double fibration. After that, Beylkin [2], Greenleaf and Uhlmann [13], Quinto [26], Gonzalez and Quinto [11] developed the study of microlocal properties, and applied to the more general support theorems, partial differential equations, etc.

Let $Z \subset \Xi \times X$ be the incidence relation, that is, $Z = \{(\xi, x) \in \Xi \times X; x \in \xi\}$, and we consider the following double fibration diagram.

$$\begin{array}{ccc} & Z & \\ \pi_{\Xi} \swarrow & & \searrow \pi_X \\ \Xi & & X \end{array}$$

Then the Schwartz kernel of \mathcal{R} is δ_Z , that is, the delta function supported on Z , and δ_Z is a Fourier integral distribution, so from Hörmander's theory Radon transform \mathcal{R} is a Fourier integral operator of order $-\dim N/2$ associated with the canonical relation $\Gamma = N^*Z \setminus 0$, the twisted conormal bundle of Z . That is,

$$\begin{aligned} \Gamma &= \{(\xi, \eta_{\xi}; x, \omega_x) \in T^*(\Xi \times X) \setminus 0; (\xi, x) \in Z, (\eta_{\xi}, -\omega_x) \perp T_{(\xi, x)} Z\} \\ &= \{(H, b, x; d_x(\lambda(A(x, b))) + d_H(\lambda(H)) - d_b(\lambda(A(x, b)))) ; \lambda \in \mathfrak{a}^* \setminus \{0\}, H = A(x, b)\}. \end{aligned}$$

Similarly, \mathcal{R}^* is a Fourier integral operator associated with the canonical relation $\Gamma^t \subset T^*X \times T^*\Xi$, which is simply Γ with (x, ω_x) and (ξ, η_{ξ}) interchanged.

Now consider the microlocal diagram

$$\begin{array}{ccc} & \Gamma & \\ \pi_{T^*\Xi} \swarrow & & \searrow \pi_{T^*X} \\ T^*\Xi & & T^*X \end{array}$$

where $\pi_{T^*\Xi}$ and π_{T^*X} again denote the natural projections. If X is real rank one, then π_Ξ becomes an injective immersion (cf. [26]), i.e. satisfies the Bolker assumption, so $\Gamma \subset T^*\Xi \times T^*X$ becomes a local canonical graph. Then by a general Fourier integral operator theory and note for Sobolev wave front set in [27], we have the following theorem.

Theorem 2.9. *Let X be a real rank one symmetric space of noncompact type. Then we have the followings:*

(i) *For any $s \in \mathbb{R}$, we have the following continuous maps:*

$$\begin{aligned} \mathcal{R} : H_{\text{comp}}^s(X) &\rightarrow H_{\text{loc}}^{s+\dim N/2}(\Xi), \\ \mathcal{R}^* : H_{\text{comp}}^s(\Xi) &\rightarrow H_{\text{loc}}^{s+\dim N/2}(X). \end{aligned}$$

(ii) *For any $u \in \mathcal{E}'(X)$, $\psi \in \mathcal{E}'(\Xi)$, and $s \in \mathbb{R}$ we have*

$$\begin{aligned} \text{WF}^{s+\dim N/2}(\mathcal{R}u) &\subset \Gamma \circ \text{WF}^s(u), \\ \text{WF}^{s+\dim N/2}(\mathcal{R}^*\psi) &\subset \Gamma^t \circ \text{WF}^s(\psi), \end{aligned}$$

where for $C \subset T^*\Xi \times T^*X$, $A \subset T^*X$, $B \subset T^*\Xi$, put

$$\begin{aligned} C \circ A &= \{(\xi; \eta_\xi) \in T^*\Xi; (\xi; \eta_\xi, x; \omega_x) \in C \text{ for some } (x; \omega_x) \in A\}, \\ C^t \circ B &= \{(x; \omega_x) \in T^*X; (x; \omega_x, \xi; \eta_\xi) \in C^t \text{ for some } (\xi; \eta_\xi) \in B\}. \end{aligned}$$

3 Basic properties of weighted L^2 -space

This section is devoted to proving local regularities of weighted L^2 -space.

We shall study the properties of weighted L^2 -space $L^{2,\delta}(X)$, by using pseudodifferential calculi on \mathfrak{a} , \mathfrak{a}^* and the Radon inversion formula.

Proposition 3.1. *For any $\delta \in \mathbb{R}$, there exists a continuous seminorm $\|\cdot\|$ on $\mathcal{S}(X)$ such that for all $u \in \mathcal{S}(X)$*

$$\|u\|_{L^{2,\delta}(X)} = \|\langle D_\lambda \rangle^\delta (\mathcal{F}u(\lambda, b) \mathbf{c}(\lambda)^{-1})\|_{L^2(\mathfrak{a}^* \times B, w^{-1} d\lambda db)} \leq \|u\|.$$

In particular, $\|u\|_{L^{2,\delta}(X)} < \infty$ for all $u \in \mathcal{D}(X)$, hence $L^{2,\delta}(X)$ is the well-defined Hilbert space.

Proof. By using Lemma 2.7 and Theorem 2.8, the assertion is obvious. \square

The basic properties of the weighted L^2 -space are the following.

Proposition 3.2. *We have*

- (i) $L^{2,0}(X) = L^2(X)$.
- (ii) $L^{2,\delta}(X) \hookrightarrow L^{2,\delta'}(X) \quad (\delta \geq \delta')$.
- (iii) For $g \in G$, put $\tau_g u(x) = u(g \cdot x)$. Then for any $\delta \in \mathbb{R}$ we have the following linear continuous map:

$$L^{2,\delta}(X) \ni u \mapsto \tau_g u \in L^{2,\delta}(X).$$

Hence $L^{2,\delta}(X)$ is a G -invariant Hilbert space.

- (iv) For any $\delta \in \mathbb{R}$, the following inclusion maps are all continuous dense embedding:

$$\mathcal{D}(X) \hookrightarrow \mathcal{S}(X) \hookrightarrow L^{2,\delta}(X).$$

- (v) By the natural coupling we have

$$L^{2,\delta}(X) \hookrightarrow \mathcal{S}'(X).$$

Proof. (i) It follows from the Plancherel theorem immediately.

(ii) Since for any $s > 0$, $\langle D_\lambda \rangle^{-s} : L^2(\mathfrak{a}^*) \rightarrow L^2(\mathfrak{a}^*)$ is continuous, it's obvious.

(iii) For any $g \in G$ and $u \in \mathcal{D}(X)$, we have

$$\mathcal{F}(\tau_g u)(\lambda, b) = e^{(i\lambda - \rho)(A(g \cdot o, g \cdot b))} \mathcal{F}u(\lambda, g \cdot b).$$

From the definition of the weighted L^2 -norm, we see that

$$\begin{aligned} & \|\tau_g u\|_{L^{2,\delta}(X)} \\ &= \|\langle D_\lambda \rangle^\delta (\mathcal{F}(\tau_g u)(\lambda, b) \mathbf{c}^{-1}(\lambda))\|_{L^2(\mathfrak{a}^* \times B, w^{-1} dHdb)} \\ &= \|\langle D_\lambda \rangle^\delta e^{(i\lambda - \rho)(A(g \cdot o, g \cdot b))} \langle D_\lambda \rangle^{-\delta} \langle D_\lambda \rangle^\delta (\mathcal{F}u(\lambda, b) \mathbf{c}^{-1}(\lambda))\|_{L^2(\mathfrak{a}^* \times B, w^{-1} dHdb)}. \end{aligned}$$

By using the basic theory of pseudodifferential operators, we can find

$$\sup_{b \in B} \|\langle D_\lambda \rangle^\delta e^{(i\lambda - \rho)(A(g \cdot o, g \cdot b))} \langle D_\lambda \rangle^{-\delta}\|_{\mathcal{L}(L^2(\mathfrak{a}^*))} < \infty,$$

where $\|\cdot\|_{\mathcal{L}(L^2(\mathfrak{a}^*))}$ is a operator norm on $L^2(\mathfrak{a}^*)$. Thus we obtain

$$\|\tau_g u\|_{L^{2,\delta}(X)} \leq \sup_{b \in B} \|\langle D_\lambda \rangle^\delta e^{(i\lambda - \rho)(A(g \cdot o, g \cdot b))} \langle D_\lambda \rangle^{-\delta}\|_{\mathcal{L}(L^2(\mathfrak{a}^*))} \|u\|_{L^{2,\delta}(X)}.$$

Since $\mathcal{D}(X)$ is dense in $L^{2,\delta}(X)$, we get the linear continuous map.

(iv) By the definition of $L^{2,\delta}(X)$ and from Proposition 3.1, it holds.

(v) We can easily see that the following Hermitian form on $\mathcal{D}(X) \times \mathcal{D}(X)$:

$$(u, v) = w^{-1} \iint_{\mathfrak{a}^* \times B} \langle D_\lambda \rangle^\delta (\mathcal{F}u(\lambda, b) \mathbf{c}^{-1}(\lambda)) \overline{\langle D_\lambda \rangle^\delta (\mathcal{F}v(\lambda, b) \mathbf{c}^{-1}(\lambda))} d\lambda db,$$

can be uniquely continuously extended to its on $L^{2,\delta}(X) \times \mathcal{S}(X)$. \square

Now, we will examine the local regularity of the weighted L^2 -space $L^{2,\delta}(X)$.

Proposition 3.3. *Suppose X is real rank one, then we have following continuous embeddings:*

$$L^2_{\text{comp}}(X) \hookrightarrow L^{2,\delta}(X) \hookrightarrow L^2_{\text{loc}}(X). \quad (3.1)$$

Proof. First, we will prove $L^2_{\text{comp}}(X) \hookrightarrow L^{2,\delta}(X)$. It is enough to show for any $R > 0$ and $k \in \mathbb{N}$ that there exists a constant $C_{k,R} > 0$ such that for all $u \in \mathcal{D}(X)$ with $\text{supp } u \subset \overline{B(o, R)}$ the following estimate holds

$$\|u\|_{L^{2,2k}(X)} \leq C_{k,R} \|u\|_{L^2(X)}.$$

Take $\chi_1 \in \mathcal{D}(\mathfrak{a})$ with $0 \leq \chi_1 \leq 1$ such that $\chi_1 = 1$ near $\{H \in \mathfrak{a}; |H| \leq R\}$. Then for any $j \in \mathbb{N}$, we have

$$\begin{aligned} & \frac{\partial^j}{\partial \lambda^j} (\mathcal{F}u(\lambda, b) \mathbf{c}^{-1}(\lambda)) \\ &= \sum_{j_1+j_2=j} \frac{j!}{j_1!j_2!} \left(\frac{\partial^{j_2}}{\partial \lambda^{j_2}} \mathbf{c}^{-1}(\lambda) \right) \int_{\mathfrak{a}} e^{-i\lambda(H)} (-iH)^{j_1} e^{\rho(H)} \mathcal{R}u(H, b) dH; \end{aligned}$$

By putting $p_{j_2}(\lambda) = \partial_{\lambda}^{j_2} \mathbf{c}^{-1}(\lambda)$, we can rewrite

$$\begin{aligned} &= \int_{\mathfrak{a}} e^{-i\lambda(H)} \sum_{j_1+j_2=j} \frac{j!}{j_1!j_2!} \bar{p}_{j_2}(D_H)^* (-iH)^{j_1} \chi_1(H) (e^{\rho(H)} \mathcal{R}u(H, b)) dH \\ &= \int_{\mathfrak{a}} e^{-i\lambda(H)} q_j(H, D_H) (e^{\rho(H)} \mathcal{R}u(H, b)) dH, \end{aligned}$$

where we set

$$q_j(H, D_H) = \sum_{j_1+j_2=j} \frac{j!}{j_1!j_2!} \bar{p}_{j_2}(D_H)^* (-iH)^{j_1} \chi_1(H).$$

Since $q_j(H, D_H) \in OpS^{\dim N/2}$, there exists a constant $C_{R,j} > 0$ such that

$$\begin{aligned} \left\| \frac{\partial^j}{\partial \lambda^j} (\mathcal{F}u(\lambda, b) \mathbf{c}^{-1}(\lambda)) \right\|_{L^2(\mathfrak{a}^* \times B, w^{-1} d\lambda db)} &\leq C_{R,j} \|e^{\rho(H)} \mathcal{R}u(H, b)\|_{L^2(B; H^{\dim N/2}(\mathfrak{a}))} \\ &= C_{R,j} \|\langle \lambda \rangle^{\dim N/2} \mathcal{F}u(\lambda, b)\|_{L^2(\mathfrak{a}^* \times B, d\lambda db)}. \end{aligned}$$

By the assumption, from Lemma 2.7 (ii), $\mathbf{c}^{-1}(\lambda)$ is the elliptic symbol of order $\dim N/2$, so there exist $R' > 0$ and $C' > 0$ such that for all $|\lambda| \geq R'$

$$|\mathbf{c}^{-1}(\lambda)| \geq C'^{-1} \langle \lambda \rangle^{\dim N/2}.$$

Take $\chi_2 \in \mathcal{D}(\mathfrak{a})$ with $0 \leq \chi_2 \leq 1$, $\text{supp } \chi_2 \subset \{H \in \mathfrak{a}; |H| \leq R' + 1\}$ and $\chi_2 = 1$ near $\{H \in \mathfrak{a}; |H| \leq R' + 1\}$, then we have

$$\begin{aligned} & |\langle \lambda \rangle^{\dim N/2} \mathcal{F}u(\lambda, b)| \\ &= \langle \lambda \rangle^{\dim N/2} \chi_2(\lambda) |\mathcal{F}u(\lambda, b)| + \langle \lambda \rangle^{\dim N/2} (1 - \chi_2(\lambda)) |\mathcal{F}u(\lambda, b)| \\ &\leq \langle R' \rangle^{\dim N/2} \chi_2(\lambda) |\mathcal{F}u(\lambda, b)| + C' |\mathbf{c}^{-1}(\lambda)| (1 - \chi_2(\lambda)) |\mathcal{F}u(\lambda, b)| \\ &\leq \langle R' \rangle^{\dim N/2} \chi_2(\lambda) |\mathcal{F}u(\lambda, b)| + C' |\mathcal{F}u(\lambda, b) \mathbf{c}^{-1}(\lambda)|. \end{aligned}$$

For the first term in above inequality, by using the Hölder inequality we find

$$\begin{aligned} |\mathcal{F}u(\lambda, b)| &= \left| \int_X e^{(-i\lambda + \rho)(A(x, b))} u(x) dx \right| \\ &\leq \int_{B(o, R)} e^{\rho(A(x, b))} |u(x)| dx \\ &\leq e^{|\rho|R} \text{Vol}(B(o, R))^{\frac{1}{2}} \|u\|_{L^2(X)}. \end{aligned}$$

By using the Plancherel formula, there exists a constant $C_{R, R', j} > 0$ such that

$$\left\| \frac{\partial^j}{\partial \lambda^j} (\mathcal{F}(\lambda, b) \mathbf{e}^{-1}(\lambda)) \right\|_{L^2(\mathfrak{a}^* \times B, w^{-1} \lambda db)} \leq C_{R, R', j} \|u\|_{L^2(X)}.$$

Since $\langle D_\lambda \rangle^{2k} = (1 - \partial_\lambda^2)^k$, we obtain desired estimates.

Next, we will prove $L^{2, \delta}(X) \hookrightarrow L^2_{\text{comp}}(X)$. For any $u \in \mathcal{D}(X)$ and $\chi_1 \in \mathcal{D}(X)$, by using the Radon inversion formula we have

$$\chi_1 u = w^{-1} \chi_1 \mathcal{R}^* \bar{\Lambda} \Lambda \mathcal{R} u.$$

Since $\text{supp } \chi_1$ is compact, there exists a sufficiently large $R > 0$ such that $\text{supp } \chi_1 \subset \overline{B(o, R)}$. We can take $\chi_2 \in \mathcal{D}(\mathfrak{a})$ with $0 \leq \chi_2 \leq 1$, $\text{supp } \chi_2 \subset \{H \in \mathfrak{a}; |H| \leq R + 1\}$ and $\chi_2 = 1$ near $\{H \in \mathfrak{a}; |H| \leq R\}$. Then we have

$$\mathcal{R}^* \chi_2 \bar{\Lambda} \Lambda \mathcal{R} u = \mathcal{R}^* \bar{\Lambda} \Lambda \mathcal{R} u$$

on $\overline{B(o, R)}$, hence we obtain

$$\chi_1 u = w^{-1} \chi_1 \mathcal{R}^* \chi_2 \bar{\Lambda} \Lambda \mathcal{R} u \quad \text{in } \mathcal{D}(X).$$

By a density argument, we can uniquely extend the linear map

$$\mathcal{D}(X) \ni u \mapsto \langle H \rangle^\delta \Lambda \mathcal{R} u \in L^2(\Xi, w^{-1} d\xi)$$

to the linear isometry operator

$$L^{2, \delta}(X) \ni u \mapsto \varphi_u \in L^2(\Xi, w^{-1} d\xi)$$

which satisfying

$$\chi_1 u = w^{-1} \chi_1 \mathcal{R}^* \chi_2 \bar{\Lambda} \langle H \rangle^{-\delta} \varphi_u \quad \text{in } \mathcal{E}'(X)$$

for any $u \in L^{2, \delta}(X)$.

Now take $\chi_3 \in \mathcal{D}(\mathfrak{a})$ such that $0 \leq \chi_3 \leq 1$ and $\chi_3 = 1$ near $\text{supp } \chi_2$, then we have $\text{supp } \chi_2 \cap \text{supp}(1 - \chi_3) = \emptyset$. Then we can decompose $\chi_1 u \in \mathcal{E}'(X)$ as

$$\begin{aligned} \chi_1 u &= w^{-1} \chi_1 \mathcal{R}^* \chi_2 \bar{\Lambda} \chi_3 \langle H \rangle^{-\delta} \varphi_u + w^{-1} \chi_1 \mathcal{R}^* \chi_2 \bar{\Lambda} (1 - \chi_3) \langle H \rangle^{-\delta} \varphi_u \\ &= u_1 + u_2. \end{aligned}$$

To estimate the first term, set

$$q(H, D_H) = \chi_2 e^{-\rho(H)} \overline{\mathbf{c}}^{-1}(D_H) \chi_3 \langle H \rangle^{-\delta} \in OpS^{\dim N/2}.$$

Then we have

$$u_1 = w^{-1} \chi_1 \mathcal{R}^* p(H, D_H) (e^{\rho(H)} \varphi_u).$$

Since all the following maps are continuous

$$\begin{aligned} p(H, D_H) : L^2(\mathfrak{a} \times B, dHdb) &\rightarrow L^2(B; H_{\text{comp}}^{-\dim N/2}(\mathfrak{a})), \\ \iota : L^2(B; H_{\text{comp}}^{-\dim N/2}(\mathfrak{a})) &\hookrightarrow H_{\text{comp}}^{-\dim N/2}(\mathfrak{a} \times B), \\ \mathcal{R}^* : H_{\text{comp}}^{-\dim N/2}(\mathfrak{a} \times B) &\rightarrow L_{\text{loc}}^2(X), \end{aligned}$$

there exists a constant $C_1 > 0$ such that

$$\begin{aligned} \|u_1\|_{L^2} &\leq C_1 \|e^{\rho(H)} \varphi_u(H, b)\|_{L^2(\mathfrak{a} \times B, w^{-1} dHdb)} \\ &\leq C_1 \|\varphi_u\|_{L^2(\Xi, w^{-1} d\xi)}. \end{aligned} \tag{3.2}$$

For the second term, set

$$r(H, D_H) = \chi_2 e^{-\rho(H)} \overline{\mathbf{c}}^{-1}(D_H) (1 - \chi_3) \langle H \rangle^{-\delta} \in OpS^{-\infty},$$

then we can write

$$u_2 = w^{-1} \chi_{-1} \mathcal{R}^* r(H, D_H) (\langle H \rangle^{-\delta} e^{\rho(H)} \varphi_u).$$

Since $r(H, D_H) \in OpS^{-\infty}$, there exists the kernel $K(H, H') \in \mathcal{E}(\mathfrak{a} \times \mathfrak{a})$ which satisfies for any $j_1, j_2, j_3 \in \mathbb{N}$

$$\sup_{(H, H') \in \mathfrak{a} \times \mathfrak{a}} |\langle H' \rangle^{-j_1} \partial_H^{j_2} \partial_{H'}^{j_3} K(H, H')| < \infty,$$

and for any $\varphi \in \mathcal{S}'(\mathfrak{a})$

$$r(H, D_H) \varphi(H) = \int_{\mathfrak{a}} K(H, H - H') \varphi(H') dH'.$$

Then the distribution kernel of $r(H, D_H) \langle H \rangle^{-\delta} = \chi_3 r(H, D_H) \langle H \rangle^{-\delta}$ is

$$\chi_3(H) K(H, H - H') \langle H' \rangle^{-\delta} \in L^2(\mathfrak{a} \times \mathfrak{a}, dH dH'),$$

i.e. the Hilbert-Schmidt kernel, so we have the following continuous map

$$r(H, D_H) \langle H \rangle^{-\delta} : L^2(\mathfrak{a} \times B, dHdb) \rightarrow L_{\text{comp}}^2(\mathfrak{a} \times B, dHdb),$$

as well as

$$\mathcal{R}^* : L_{\text{comp}}^2(\Xi) \rightarrow H_{\text{loc}}^{\dim N/2}(X) \hookrightarrow L_{\text{loc}}^2(X).$$

Hence there exists a constant $C_2 > 0$ such that

$$\begin{aligned} \|u_2\|_{L^2} &\leq C_2 \|e^{\rho(H)} \varphi_u(H, b)\|_{L^2(\mathfrak{a} \times B, w^{-1} dHdb)} \\ &\leq C_2 \|\varphi_u\|_{L^2(\Xi, w^{-1} d\xi)} \end{aligned} \tag{3.3}$$

Thus from (3.2), (3.3), we obtain $\chi_1 u \in L^2(X)$ and there exists a constant $C_R > 0$ such that

$$\|\chi_1 u\|_{L^2(X)} \leq C_R \|\varphi_u\|_{L^2(\Xi, w^{-1} d\xi)} = C_R \|u\|_{L^{2,\delta}(X)}.$$

So we complete the proof. \square

4 Time-global smoothing effects

In this section, we prove Theorem 1.2, that is, a smoothing effect of radially symmetric constant coefficients pseudodifferential equations on real rank one symmetric spaces of noncompact type. The main idea of the proof is to reduce the argument to the Euclidean case by introducing some isometry operator from L^2 space on a Riemannian symmetric space to its on the horocycle space. We will see later this isometry transform a solution on the Riemannian symmetric space to the solution of the “same” equation with respect to the Euclidean variable on the horocycle space.

Definition 4.1. For $s \in W$, define $T_s : L^2(X) \rightarrow L^2(\mathfrak{a} \times B, dHdb)$ by

$$T_s u(H, b) = \int_{s\mathfrak{a}_+^*} e^{i\lambda(H)} \mathcal{F}u(\lambda, b) \mathbf{c}^{-1}(\lambda) d\lambda$$

and define $T : L^2(X) \rightarrow L^2(\mathfrak{a} \times B, w^{-1}dHdb)$ by

$$Tu(H, b) = \int_{\mathfrak{a}^*} e^{i\lambda(H)} \mathcal{F}u(\lambda, b) \mathbf{c}^{-1}(\lambda) d\lambda.$$

Proposition 4.2. Both T_s and T are linear isometry operators respectively.

Proof. It follows immediately from the W -invariantness of Fourier images of L^2 functions and the Plancherel theorem. \square

Proof of Theorem 1.2 (i) For any $s \in W$, we have

$$\begin{aligned} T_s(p(D_x)e^{ita(D_x)}\psi)(H, b) &= \int_{s\mathfrak{a}_+^*} e^{i\lambda(H)} \mathcal{F}(p(D_x)e^{ita(D_x)}\psi)(\lambda) \mathbf{c}^{-1}(\lambda) d\lambda \\ &= \int_{s\mathfrak{a}_+^*} e^{i\lambda(H)} p(s\lambda) e^{ita(s\lambda)} \mathcal{F}\psi(\lambda, b) \mathbf{c}^{-1}(\lambda) d\lambda \\ &= p(sD_H) e^{ita(sD_H)} (T_s\psi)(H, b). \end{aligned}$$

After decomposing \mathfrak{a}^* as the union of $s\mathfrak{a}_+^*$ ($s = \pm 1$), we see that

$$\begin{aligned} &\langle H \rangle^{-\delta} T(p(D_x)e^{ita(D_x)}\psi)(H, b) \\ &= \sum_{s \in W} \langle H \rangle^{-\delta} p(sD_H) e^{ita(sD_H)} (T_s\psi)(H, b). \end{aligned}$$

Then by taking L^2 -norm over $\mathbb{R} \times \mathfrak{a} \times B$ and applying the assumption, we obtain

$$\begin{aligned} &\|p(D_x)e^{ita(D_x)}\psi\|_{L^2(\mathbb{R}; L^{2, -\delta}(X))} \\ &\leq \sum_{s \in W} \|\langle H \rangle^{-\delta} p(sD_H) e^{ita(sD_H)} (T_s\psi)(H, b)\|_{L^2(\mathbb{R} \times \mathfrak{a} \times B, w^{-1}dt dHdb)} \\ &\leq w^{1/2} C_\delta \|\psi\|_{L^2(X)}. \end{aligned}$$

(ii) For χ , we can take a cut off function $\chi_1 \in C^\infty(\mathfrak{a})$ such that $\chi_1(H) = 0$ ($|H| \leq 1/2$), $\chi_1 = 1$ on $\text{supp } \chi$, then $\chi = \chi\chi_1$ and we have

$$\begin{aligned} & T \left(\chi(D_x) q(D_x) \int_0^t e^{i(t-\tau)a(D_x)} f(\tau) d\tau \right) (H, b) \\ &= \sum_{s \in W} \chi(sD_H) q(sD_H) \int_0^t e^{i(t-\tau)a(D_H)} T_s(\chi_1(D_x) f_\tau) (H, b) d\tau. \end{aligned}$$

Then by applying the assumption, we obtain

$$\begin{aligned} & \left\| \chi(D_x) q(D_x) \int_0^t e^{i(t-\tau)a(D_H)} f_\tau d\tau \right\|_{L^2(\mathbb{R}; L^2, -\delta(X))} \\ &= \sum_{s \in \{\pm 1\}} \left\| \langle H \rangle^{-\delta} \chi(sD_H) q(sD_H) \int_0^t e^{i(t-\tau)a(sD_H)} T_s(\chi_1(D_x) f_\tau) (H, b) d\tau \right\|_{L^2(\mathbb{R} \times \mathfrak{a} \times B, w^{-1} dt dH db)} \\ &\leq \sum_{s \in \{\pm 1\}} C_\delta \left\| \langle H \rangle^\delta T_s(\chi_1(D_x) f_t) \right\|_{L^2(\mathbb{R} \times \mathfrak{a} \times B, w^{-1} dt dH db)}. \end{aligned}$$

Here we put $\chi_2(\lambda) = \chi_{\mathfrak{a}_+^*}(\lambda) \chi_1(\lambda) \in C^\infty(\mathfrak{a}^*)$, then we have

$$\langle H \rangle^\delta T_s(\chi_1(D_x) f_t) = \langle H \rangle^\delta \chi_2(D_H) \langle H \rangle^{-\delta} (\langle H \rangle^\delta T(f_t)).$$

Since $\langle H \rangle^\delta \chi_2(D_H) \langle H \rangle^{-\delta}$ is an L^2 -bounded operator on \mathfrak{a} , we obtain

$$\begin{aligned} & \left\| \chi(D_x) q(D_x) \int_0^t e^{i(t-\tau)a(D_H)} f_\tau d\tau \right\|_{L^2(\mathbb{R}; L^2, -\delta(X))} \\ &\leq C_{\delta, \chi_2} \left\| \langle H \rangle^\delta T(f_t) \right\|_{L^2(\mathbb{R} \times \mathfrak{a} \times B, w^{-1} dt dH db)} \\ &= C_{\delta, \chi_2} \|f\|_{L^2(\mathbb{R}; L^2, \delta(X))}. \end{aligned}$$

So we complete the proof. \square

5 Gain of regularity for the Schrödinger evolution equation

In this section, we prove a gain of regularity for the Schrödinger evolution equation on real rank one symmetric spaces of noncompact type. We also use the linear isometry introduced in the previous section to reduce the argument to the Euclidean case. We can obtain a gain of regularity with respect to the Euclidean variable then we can recover the regularity of the solution from its image of Radon transform by treating dual Radon transform as an elliptic Fourier integral operator.

The following estimates on one-dimensional Euclidean spaces are well known.

Proposition 5.1. *On one-dimensional Euclidean space \mathbb{R}^1 , for any $k \in \mathbb{N}$ and $\delta > 1/2$, we have the following continuous maps:*

$$\langle x \rangle^{-k} L^2(\mathbb{R}^1) \ni \phi \mapsto t^k \langle x \rangle^{-k-\delta} \langle D_x \rangle^{k+1/2} e^{-it\Delta_{\mathbb{R}^1}} \phi \in L_{\text{loc}}^2(\mathbb{R}; L^2(\mathbb{R}^1)), \quad (5.1)$$

$$\langle x \rangle^{-k} L^2(\mathbb{R}^1) \ni \phi \mapsto t^k \langle x \rangle^{-k} \langle D_x \rangle^k e^{-it\Delta_{\mathbb{R}^1}} \phi \in C(\mathbb{R}; L^2(\mathbb{R}^1)). \quad (5.2)$$

We can also translate above estimates on the one-dimensinal Euclidean space into its on symmetric spaces by using isometry T .

Proof of Theorem 1.5 For any $\varphi \in L^2(X)$ with

$$\|\langle D_\lambda \rangle^k (\mathcal{F}\varphi(\lambda, b)\mathbf{c}^{-1}(\lambda))\|_{L^2(\mathfrak{a}^* \times \Theta, d\lambda db)} < \infty,$$

where Θ is an open subset of B , we can find

$$\begin{aligned} T(e^{-it\Delta_X}\varphi)(H, b) &= e^{\rho(H)}\Lambda\mathcal{R}(e^{-it\Delta_X}\varphi)(H, b) \\ &= e^{it|\rho|^2}e^{-it\Delta_{\mathfrak{a}}}(T\varphi)(H, b) \end{aligned}$$

and

$$\begin{aligned} \|\langle H \rangle^k (T\varphi)(H, b)\|_{L^2(\mathfrak{a} \times \Theta, dHdb)} &= \|\langle D_\lambda \rangle^k (\mathcal{F}\varphi(\lambda, b)\mathbf{c}^{-1}(\lambda))\|_{L^2(\mathfrak{a}^* \times \Theta, d\lambda db)} \\ &< \infty. \end{aligned}$$

Therefore by using Proposition 5.1 we have,

$$\phi \in \langle x \rangle^{-k}L^2(\mathbb{R}^1) \Rightarrow e^{-it\Delta_{\mathbb{R}^1}}\phi \in H_{\text{loc}}^{k+1/2}(\mathbb{R}^1) \quad \text{for a.e. } t \neq 0,$$

hence we get

$$\Lambda\mathcal{R}(e^{-it\Delta_X}\varphi) \in L^2(\Theta; H_{\text{loc}}^{k+1/2}(\mathfrak{a})) \quad \text{for a.e. } t \neq 0. \quad (5.3)$$

Now take $\chi_1 \in \mathcal{D}(X)$ with $\text{supp } \chi_1 \subset \overline{B(o, 2)}$, $\chi_1 = 1$ near $B(o, 1)$, and $\chi_2 \in \mathcal{D}(\mathfrak{a})$ with $\chi_2 = 1$ near $\{H \in \mathfrak{a}; |H| \leq 1\}$, then we can write

$$\chi_1 e^{-it\Delta_X}\varphi = \chi_1 \mathcal{R}^* \chi_2 \bar{\Lambda}(\Lambda\mathcal{R}(e^{-it\Delta_X}\varphi)).$$

For $(o; \omega_b(o)) \in T_o^*X \setminus 0$ with $b \in \Theta \cap (-\text{id}_B)\Theta$, by using Theorem 2.9 we have

$$\begin{aligned} (o; \pm\omega_b(o)) &\notin \Gamma^t \circ \text{WF}^{k+1/2}(\Lambda\mathcal{R}(e^{-it\Delta_X}\varphi)) \\ \Rightarrow (o; \pm\omega_b(o)) &\notin \Gamma^t \circ \text{WF}^{k+1/2-\dim N/2}(\bar{\Lambda}(\Lambda\mathcal{R}(e^{-it\Delta_X}\varphi))) \\ \Rightarrow (o; \pm\omega_b(o)) &\notin \text{WF}^{k+1/2}(\mathcal{R}^* \chi_2 \bar{\Lambda}(\Lambda\mathcal{R}(e^{-it\Delta_X}\varphi))) \\ \Leftrightarrow (o; \pm\omega_b(o)) &\notin \text{WF}^{k+1/2}(e^{-it\Delta_X}\varphi). \end{aligned}$$

Since

$$\Gamma^t \cap T_{(o, b, 0)}^*(\Xi \times X) = \{(o, b, 0; \lambda(\omega_b(o) + dH)); \lambda \in \mathbb{R} \setminus \{0\}\},$$

we obtain

$$\begin{aligned} (o; \pm\omega_b(o)) &\notin \Gamma^t \circ \text{WF}^{k+1/2}(\Lambda\mathcal{R}(e^{-it\Delta_X}\varphi)) \\ \Leftrightarrow (o, \pm b; dH) &\notin \text{WF}^{k+1/2}(\Lambda\mathcal{R}(e^{-it\Delta_X}\varphi)). \end{aligned}$$

So from (5.3), we get $(o; \pm\omega_b(o)) \notin \text{WF}^{k+1/2}(e^{-it\Delta_X}\varphi)$ for a.e. $t \neq 0$.

Next, by using G -invariantness of the Schrödinger evolution group $e^{-it\Delta_X}$ we show at other points. For any $g \in G$, we have

$$\mathcal{F}(\tau_g \varphi)(\lambda, b) = e^{(i\lambda - \rho)(A(g \cdot o, g \cdot b))} \mathcal{F}\varphi(\lambda, g \cdot b),$$

where $\tau_g \varphi(x) = \varphi(g \cdot x)$. For the non-Euclidean metric $A(x, b)$ we have the following formula for the G -action:

$$A(g \cdot x, g \cdot b) = A(x, b) + A(g \cdot o, g \cdot b).$$

Also for the action on $B = G/MAN$ of G , we have $d(g \cdot b) = e^{-2\rho(A(g \cdot o, g \cdot b))} db$. Then for $\tau_g \varphi$, we see that

$$\begin{aligned} & \left\| \langle D_\lambda \rangle^k (\mathcal{F}(\tau_g \varphi)(\lambda, b) \mathbf{c}^{-1}(\lambda)) \right\|_{L^2(\mathfrak{a}^* \times g^{-1} \cdot \Theta, d\lambda db)} \\ &= \left\| \langle D_\lambda \rangle^k (e^{(i\lambda - \rho)(A(g \cdot o, g \cdot b))} \mathcal{F}\varphi(\lambda, g \cdot b) \mathbf{c}^{-1}(\lambda)) \right\|_{L^2(\mathfrak{a}^* \times g^{-1} \cdot \Theta, d\lambda db)} \\ &= \left\| \langle D_\lambda \rangle^k (e^{(i\lambda - \rho)(A(g \cdot o, b))} \mathcal{F}\varphi(\lambda, b) \mathbf{c}^{-1}(\lambda)) e^{-\rho(A(g^{-1} \cdot o, g^{-1} \cdot b))} \right\|_{L^2(\mathfrak{a}^* \times \Theta, d\lambda db)}, \end{aligned}$$

by using $A(g^{-1} \cdot o, g^{-1} \cdot b) = -A(g \cdot o, b)$, we get

$$\begin{aligned} &= \left\| \langle D_\lambda \rangle^k (e^{i\lambda(A(g \cdot o, b))} \mathcal{F}\varphi(\lambda, b) \mathbf{c}^{-1}(\lambda)) \right\|_{L^2(\mathfrak{a}^* \times \Theta, d\lambda db)} \\ &= \left\| (\langle D_\lambda \rangle^k e^{i\lambda(A(g \cdot o, b))} \langle D_\lambda \rangle^{-k}) \langle D_\lambda \rangle^k (\mathcal{F}\varphi(\lambda, b) \mathbf{c}^{-1}(\lambda)) \right\|_{L^2(\mathfrak{a}^* \times \Theta, d\lambda db)}. \end{aligned}$$

Since $\{\langle D_\lambda \rangle^k e^{i\lambda(A(g \cdot o, b))} \langle D_\lambda \rangle^{-k}\}_{b \in B} \in \mathcal{L}(L^2(\mathfrak{a}^*))$ is a bounded subset, so we get

$$\left\| \langle D_\lambda \rangle^k (\mathcal{F}(\tau_g \varphi)(\lambda, b) \mathbf{c}^{-1}(\lambda)) \right\|_{L^2(\mathfrak{a}^* \times g^{-1} \Theta, d\lambda db)} < \infty.$$

Hence $\tau_g \varphi \in L^2(X)$ and $g^{-1} \Theta \subset B$ satisfies the assumption (1.5). By applying the above argument at the origin o , for any $b \in g^{-1} \Theta \cap (-\text{id}_B)g^{-1} \Theta$, and a.e. $t \neq 0$ we obtain

$$\begin{aligned} (o; \pm \omega_b(o)) &\notin \text{WF}^{k+1/2}(e^{-it\Delta_X}(\tau_g \varphi)) \\ &= \text{WF}^{k+1/2}(\tau_g(e^{-it\Delta_X} \varphi)) \\ &= (\tau_g)^*(\text{WF}^{k+1/2}(e^{-it\Delta_X} \varphi)), \end{aligned}$$

hence

$$\begin{aligned} (g \cdot o; \pm \omega_{g \cdot b}(g \cdot o)) &= (\tau_{g^{-1}})^*((o; \omega_b(o))) \\ &\notin \text{WF}^{k+1/2}(e^{-it\Delta_X} \varphi). \end{aligned}$$

Finally, if $\varphi \in L^{2,\delta}(X)$, then we can take $\Theta = B$, so the assertion folds. For gain of H_{loc}^k -regularity, we can also prove by the same argument. So we complete the proof. \square

Finally, we prove Theorem 1.6.

Proof of Theorem 1.6 For any $\varphi \in L^{2,k}(X)$, from the definition of the weighted L^2 -norm, we have

$$\begin{aligned} & \|t^k \langle D_x \rangle^{k+1/2} e^{-it\Delta_X} \varphi\|_{L^{2,-k-\delta}(X)} \\ &= \|t^k \langle H \rangle^{-k-\delta} T \left(\langle D_x \rangle^{k+1/2} e^{-it\Delta_X} \varphi \right)\|_{L^2(\mathfrak{a} \times B, w^{-1} dHdb)} \\ &= \|t^k \langle H \rangle^{-k-\delta} \langle D_H \rangle^{k+1/2} e^{-it\Delta_{\mathfrak{a}} + it|\rho|^2} T\varphi\|_{L^2(\mathfrak{a} \times B, w^{-1} dHdb)}. \end{aligned}$$

For any $T > 0$, by using the continuity of the map (5.1), we obtain

$$\begin{aligned} & \|t^k \langle D_x \rangle^{k+1/2} e^{-it\Delta_X} \varphi\|_{L^2((-T,T); L^{2,-k-\delta}(X))} \\ &= \|t^k \langle H \rangle^{-k-\delta} \langle D_H \rangle^{k+1/2} e^{-it\Delta_{\mathfrak{a}}} (T\varphi)\|_{L^2((-T,T) \times \mathfrak{a} \times B; w^{-1} dHdb)} \\ &\leq C_T \| \langle H \rangle^k (T\varphi) \|_{L^2(\mathfrak{a} \times B; w^{-1} dHdb)} \\ &= C_T \|\varphi\|_{L^{2,k}(X)}, \end{aligned}$$

for some $C_T > 0$ depending on T . The rest assertion proved by the same argument. \square

References

- [1] M. Ben-Artzi and S. Klainerman, *Decay and regularity for the Schrödinger equation*, J. Anal. Math. **58** (1992), 25–37.
- [2] G. Beylkin, *The inversion problem and applications of the generalized Radon transform*, Comm. Pure Appl. Math. **37** (1984), no. 5, 579–599.
- [3] H. Chihara, *Smoothing effects of dispersive pseudodifferential equations*, Comm. Partial Differential Equations **27** (2002), no. 9-10, 1953–2005.
- [4] W. Craig, T. Kappeler, and W. Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm. Pure Appl. Math. **48** (1995), no. 8, 769–860.
- [5] S.-i. Doi, *Remarks on the Cauchy problem for Schrödinger-type equations*, Comm. Partial Differential Equations **21** (1996), no. 1-2, 163–178.
- [6] ———, *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds*, Duke Math. J. **82** (1996), no. 3, 679–706.
- [7] ———, *Commutator algebra and abstract smoothing effect*, J. Funct. Anal. **168** (1999), no. 2, 428–469.
- [8] ———, *Smoothing effects for Schrödinger evolution equation and global behavior of geodesic flow*, Math. Ann. **318** (2000), no. 2, 355–389.
- [9] M. Eguchi and K. Okamoto, *The Fourier transform of the Schwartz space on a symmetric space*, Proc. Japan Acad. Ser. A Math. Sci. **53** (1977), no. 7, 237–241.
- [10] R. Gangolli and V. S. Varadarajan, “Harmonic analysis of spherical functions on real reductive groups”, *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*, vol. 101, Springer-Verlag, Berlin, 1988.
- [11] F. Gonzalez and E. T. Quinto, *Support theorems for Radon transforms on higher rank symmetric spaces*, Proc. Amer. Math. Soc. **122** (1994), no. 4, 1045–1052.

- [12] V. Guillemin and S. Sternberg, “Geometric asymptotics”, American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.
- [13] A. Greenleaf and G. Uhlmann, *Microlocal techniques in integral geometry*, Integral geometry and tomography (Arcata, CA, 1989), Contemp. Math., vol. 113, Amer. Math. Soc., Providence, RI, 1990, pp. 121–135.
- [14] Harish-Chandra, *Discrete series for semisimple Lie groups. II. Explicit determination of the characters*, Acta Math. **116** (1966), 1–111.
- [15] S. Helgason, *Fundamental solutions of invariant differential operators on symmetric spaces*, Amer. J. Math. **86** (1964), 565–601.
- [16] ———, *A duality for symmetric spaces with applications to group representations*, Advances in Math. **5** (1970), 1–154.
- [17] ———, *Paley-Wiener theorems and surjectivity of invariant differential operators on symmetric spaces and Lie groups*, Bull. Amer. Math. Soc. **79** (1973), 129–132.
- [18] ———, “Groups and geometric analysis”, Pure and Applied Mathematics, vol. 113, Academic Press Inc., Orlando, FL, 1984. Integral geometry, invariant differential operators, and spherical functions.
- [19] ———, “Geometric analysis on symmetric spaces”, Mathematical Surveys and Monographs, vol. 39, American Mathematical Society, Providence, RI, 1994.
- [20] ———, “Differential geometry, Lie groups, and symmetric spaces”, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.
- [21] L. Hörmander, “The analysis of linear partial differential operators. III”, Vol. 274, Springer-Verlag, Berlin, 1985.
- [22] T. Hoshiro, *Decay and regularity for dispersive equations with constant coefficients*, J. Anal. Math. **91** (2003), 211–230.
- [23] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in applied mathematics, Adv. Math. Suppl. Stud., vol. 8, Academic Press, New York, 1983, pp. 93–128.
- [24] T. Kato and K. Yajima, *Some examples of smooth operators and the associated smoothing effect*, Rev. Math. Phys. **1** (1989), no. 4, 481–496.
- [25] H. Kumano-go, “Pseudodifferential operators”, MIT Press, Cambridge, Mass., 1981.
- [26] E. T. Quinto, *Real analytic Radon transforms on rank one symmetric spaces*, Proc. Amer. Math. Soc. **117** (1993), no. 1, 179–186.
- [27] ———, *An introduction to X-ray tomography and Radon transforms*, The Radon transform, inverse problems, and tomography, Proc. Sympos. Appl. Math., vol. 63, Amer. Math. Soc., Providence, RI, 2006, pp. 1–23.
- [28] I. Rodnianski and T. Tao, *Longtime decay estimates for the Schrödinger equation on manifolds*, Mathematical aspects of nonlinear dispersive equations, Ann. of Math. Stud., vol. 163, Princeton Univ. Press, Princeton, NJ, 2007, pp. 223–253.
- [29] M. A. Shubin, “Pseudodifferential operators and spectral theory”, 2nd ed., Springer-Verlag, Berlin, 2001.
- [30] M. Sugimoto, *Global smoothing properties of generalized Schrödinger equations*, J. Anal. Math. **76** (1998), 191–204.
- [31] J. Tits, *Sur certaines classes d’espaces homogènes de groupes de Lie*, Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **29** (1955), no. 3, 268 (French).

MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI 980-8578
JAPAN

E-mail address: sa6m13@math.tohoku.ac.jp